Cut-by-curves criterion for the overconvergence of p-adic differential equations

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Abstract

In this paper, we prove a 'cut-by-curves criterion' for the overconvergence of integrable connections on certain rigid analytic spaces and certain varieties over p-adic fields.

Contents

Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers O_K and residue field k. Assume we are given an open immersion $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ of p-adic formal schemes separated and smooth over $\mathrm{Spf}\, O_K$ such that the complement is a relative simple normal crossing divisor and let $X \hookrightarrow \overline{X}$ be the special fiber of $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$. (In this paper, we call such a pair $(\mathcal{X}, \overline{\mathcal{X}})$ *a formal smooth pair* and call the pair $(X, \overline{X})$ *the special fiber* of $(\mathcal{X}, \overline{\mathcal{X}})$.) Then we have an admissible open immersion $\mathcal{X}_K \hookrightarrow \overline{\mathcal{X}}_K$ of associated rigid spaces over K and an open immersion $]\overline{X} - X[_{\overline{\mathcal{X}}} \hookrightarrow \overline{\mathcal{X}}_K$ from the tubular neighborhood of $\overline{X} - X$ in $\overline{X}_K$. A strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ is an admissible open set $\mathfrak{U}$ of $\overline{\mathcal{X}}_K$ containing $\mathcal{X}_K$ such that $\{\mathfrak{U},]\overline{X} - X[_{\overline{\mathcal{X}}}\}$ forms an admissible covering of $\overline{\mathcal{X}}_K$. Let $\mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K)$ be the category of pairs $(\mathfrak{U}, (E, \nabla))$ consisting of a strict neighborhood $\mathfrak{U}$ of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ and a ∇-module (=locally free module of finite rank endowed with an integrable connection) (E, ∇) on $\mathfrak{U}$ over K, whose set of morphisms is defined by $\mathrm{Hom}((\mathfrak{U}, (E, \nabla)), (\mathfrak{U}', (E', \nabla'))) := \varinjlim_{\mathfrak{U}''} \mathrm{Hom}((E, \nabla)|_{\mathfrak{U}''}, (E', \nabla')|_{\mathfrak{U}''})$, where $\mathfrak{U}''$ runs

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through strict neighborhoods of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ contained in $\mathfrak{U}\cap\mathfrak{U}'$. We call an object in $\mathrm{MIC}((\mathcal{X}_K,\overline{\mathcal{X}}_K)/K)$ a ∇-module on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ by abuse of terminology, and we will often denote it simply by (E,∇) in the following. We say that a ∇-module on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ is *overconvergent* if it comes from an overconvergent isocrystal on $(X,\overline{X})/K$. In this paper, as the first main theorem, we prove a 'cut-by-curves criterion' for the overconvergence of integrable connections on strict neighborhoods of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$.

A morphism $f:(\mathcal{Y},\overline{\mathcal{Y}})\longrightarrow(\mathcal{X},\overline{\mathcal{X}})$ of formal smooth pairs is a morphism $f:\overline{\mathcal{Y}}\longrightarrow\overline{\mathcal{X}}$ satisfying $f(\mathcal{Y})\subseteq\mathcal{X}$ and it is called strict if $f^{-1}(\mathcal{X})=\mathcal{Y}$. f is called a (locally) closed immersion if so is the morphism $f:\overline{\mathcal{Y}}\longrightarrow\overline{\mathcal{X}}$. If $f:(\mathcal{Y},\overline{\mathcal{Y}})\longrightarrow(\mathcal{X},\overline{\mathcal{X}})$ is a morphism of formal smooth pairs and if (E,∇) is a ∇-mobule on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$, we can define in natural way the pull-back $f^*(E,\nabla)$ of (E,∇) by f, which is a ∇-module on a strict neighborhood of $\mathcal{Y}_K$ in $\overline{\mathcal{Y}}_K$. Then our first main theorem is described as follows:

Theorem 0.1. *Let K,k be as above and assume that k is uncountable. Let $(\mathcal{X},\overline{\mathcal{X}})$ be a formal smooth pair and let (E,∇) be a ∇-module on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$. Then the following are equivalent*:

(1) *(E,∇) is overconvergent.*

(2) *For any strict locally closed immersion $i:(\mathcal{Y},\overline{\mathcal{Y}})\hookrightarrow(\mathcal{X},\overline{\mathcal{X}})$ of formal smooth pairs with $\dim\overline{\mathcal{Y}}=1$, $i^*(E,\nabla)$ is overconvergent.*

Note that, in Theorem 0.1, the implication $(1)\Longrightarrow(2)$ is easy: Indeed, if we denote the morphism induced by i on special fibers by $i_0:(Y,\overline{Y})\longrightarrow(X,\overline{X})$ and if (E,∇) comes from an overconvergent isocrystal $\mathcal{E}$ on $(X,\overline{X})/K$, $i^*(E,\nabla)$ comes from the overconvergent isocrystal $i_0^*\mathcal{E}$ on $(Y,\overline{Y})/K$ and hence it is overconvergent. So what we should prove is the implication $(2)\Longrightarrow(1)$. We prove it by using Kedlaya's result ([Ke1]) on etale covers of smooth k-varieties, the notion of intrinsic generic radius of convergence of ∇-modules on polyannuli due to Kedlaya-Xiao ([K-X]) and some techniques developped in [S].

The second main theorem is an algebraic variant of the theorem above. Assume we are given an open immersion $\mathbf{X}\hookrightarrow\overline{\mathbf{X}}$ of smooth schemes over $\operatorname{Spec}O_K$ such that the complement is a relative simple normal crossing divisor. (In this paper, we call such a pair $(\mathbf{X},\overline{\mathbf{X}})$ *a smooth pair over O_K*.) Denote the generic fiber of $\mathbf{X}$ by $\mathbf{X}_K$ and let $\mathrm{MIC}((\mathbf{X}_K/K)$ be the category of ∇-module (=locally free module of finite rank endowed with an integrable connection) (E,∇) on $\mathbf{X}_K$ over K in algebraic sense. Let $\mathbf{X}_K^{\mathrm{an}}$ be the rigid analytic space associated to the K-scheme $\mathbf{X}_K$ and let $\mathrm{MIC}((\mathbf{X}_K^{\mathrm{an}}/K)$ be the category of ∇-modules on $\mathbf{X}_K^{\mathrm{an}}$ over K in analytic sense. Also, let $\widehat{\mathbf{X}},\widehat{\overline{\mathbf{X}}}$ be the p-adic completion of $\mathbf{X},\overline{\mathbf{X}}$, respectively. Then $\mathbf{X}_K^{\mathrm{an}}\cap\widehat{\overline{\mathbf{X}}}_K$ is a strict neighborhhod of $\widehat{\mathbf{X}}_K$ in $\widehat{\overline{\mathbf{X}}}_K$. Hence we have the functors

$$\begin{array}{ccccc}\mathrm{MIC}(\mathbf{X}_K/K) & \longrightarrow & \mathrm{MIC}(\mathbf{X}_K^{\mathrm{an}}/K) & \longrightarrow & \mathrm{MIC}((\widehat{\mathbf{X}}_K,\widehat{\overline{\mathbf{X}}}_K)/K); \\ (E,\nabla) & \longmapsto & (E^{\mathrm{an}},\nabla^{\mathrm{an}}) & \longmapsto & (\mathbf{X}_K^{\mathrm{an}}\cap\widehat{\overline{\mathbf{X}}}_K,(E^{\mathrm{an}},\nabla^{\mathrm{an}})),\end{array}$$

where the first one is the analytification. We call an object (E, ∇) in $\mathrm{MIC}(\mathbf{X}_K/K)$ *overconvergent* if the associated object $(\mathbf{X}_K^{\mathrm{an}} \cap \widehat{\overline{\mathbf{X}}}_K, (E^{\mathrm{an}}, \nabla^{\mathrm{an}}))$ in $\mathrm{MIC}((\widehat{\mathbf{X}}_K, \widehat{\overline{\mathbf{X}}}_K)/K)$ is overconvergent.

We can define the notion of a (strict) morphism $f : (\mathbf{Y}, \overline{\mathbf{Y}}) \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ of smooth pairs over O_K in natural way and it is called a (locally) closed immersion if so is the morphism $f : \overline{\mathbf{Y}} \longrightarrow \overline{\mathbf{X}}$. As in the case of formal smooth pairs, we can define the pull-back of a ∇-module on $\mathbf{X}_K$ by f, which is a ∇-module on $\mathbf{Y}_K$. Then our second main theorem is described as follows:

Theorem 0.2. *Let K, k be as above and assume that k is uncountable. Let $(\mathbf{X}, \overline{\mathbf{X}})$ be a formal smooth pair such that $\overline{\mathbf{X}}$ is projective over O_K and let (E, ∇) be a ∇-module on $\mathbf{X}_K$. Then the following are equivalent*:

(1) (E, ∇) *is overconvergent.*

(2) *For any strict locally closed immersion $i : (\mathbf{Y}, \overline{\mathbf{Y}}) \hookrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ of smooth pairs over O_K with* $\dim(\overline{\mathbf{Y}}/O_K) = 1$, $i^*(E, \nabla)$ *is overconvergent.*

Since the implication $(1) \Longrightarrow (2)$ is obvious as in the case of Theorem 0.1, it suffices to prove the implication $(2) \Longrightarrow (1)$. We prove it by reducing to a refined version of Theorem 0.1 (see Remark 1.9): To do this, we prove a partial generalization of Kedlaya's result ([Ke1, Theorem 2]) on etale covers of smooth k-varieties to the case of smooth formal schemes and smooth schemes over O_K.

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Convention

Throughout this paper, K is a complete discrete valuation field of mixed characteristic $(0, p)$ with ring of integers O_K and residue field k. Let $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$ be a fixed valuation of K and let Γ^* be $\sqrt{|K^\times|} \cup \{0\}$. For a p-adic formal scheme $\mathcal{X}$ topologically of finite type over O_K, we denote the associated rigid space by $\mathcal{X}_K$. A k-variety means a reduced separated scheme of finite type over k. A closed point in a smooth k-variety X is called a separable closed point if its residue field is separable over k. For a p-adic smooth formal scheme $\mathcal{X}$ over $\mathrm{Spf}\, O_K$ (resp. a smooth scheme $\mathbf{X}$ over $\mathrm{Spec}\, O_K$) with special fiber X and a separable closed point x of X, a lift of x in $\mathcal{X}$ (resp. $\mathbf{X}$) is a closed sub p-adic formal scheme $\widetilde{x} \hookrightarrow \mathcal{X}$ which is etale over $\mathrm{Spf}\, O_K$ (resp. a closed subscheme $\widetilde{x} \hookrightarrow \mathbf{X}$ which is etale over $\mathrm{Spec}\, O_K$) with special fiber x. (Note that there always exists a lift of x.)

We use freely the notion concerning overconvergent isocrystals. For detail, see [Be] and [Ke2, §2]. See also Propositions 1.1, 1.3 in the text.

1 Proof of the first main theorem

In this section, we give a proof of the first main theorem (Theorem 0.1). First, let us recall the basic definition concerning overconvergence and recall a concrete description of overconergence for ∇-modules, which is due to Berthelot ([Be, 2.2.13], [Ke2, 2.5.6–8]).

For a formal smooth pair $(\mathcal{X}, \overline{\mathcal{X}})$ with special fiber $(X, \overline{X})$, let us denote the category of overconvergent isocrystals on $(X, \overline{X})/K$ by $I^\dagger((X, \overline{X})/K)$. Then we have a fully-faithful functor

$$\Phi_{(\mathcal{X},\overline{\mathcal{X}})} : I^\dagger((X, \overline{X})/K) \longrightarrow \mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K)$$

which is functorial with respect to $(\mathcal{X}, \overline{\mathcal{X}})$. As is written in the introduction, we say that an object (E, ∇) in $\mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K)$ is overconvergent if it is in the essential image of $\Phi_{(\mathcal{X},\overline{\mathcal{X}})}$.

In this paper, we need a concrete description of overconergence for the objects in $\mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K)$, using the Taylor series. So we will recall it. Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair and assume, for the moment, the following condition $(*)$:

$(*)$: $\overline{\mathcal{X}}$ is affine, $\overline{\mathcal{X}} - \mathcal{X}$ is a union of smooth divisors $\mathcal{D}_i$ $(1 \leq i \leq n)$ which are defined as the zero locus of some $t_i \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ $(1 \leq i \leq m)$, and $\Omega^1_{\overline{\mathcal{X}}_K}$ is freely generated by dt_i $(1 \leq i \leq n+m)$ for $t_1, ..., t_n$ as before and some $t_{n+1}, ..., t_{n+m} \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$.

With the assumption of $(*)$, we denote the induced differential operator $\partial/\partial t_i$ (which operates on ∇-modules on strict neighborhoods of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$) simply by ∂_i. Also, for $\lambda \in (0, 1] \cap \Gamma^*$, we put

$$\mathfrak{U}_\lambda := \{x \in \overline{\mathcal{X}}_K \,|\, |t_i(x)| \geq \lambda \text{ for all } 1 \leq i \leq n\}.$$

Then we can give a concrete description of overconergence in terms of Taylor series in the following way:

Proposition 1.1 ([Be, 2.2.13], [Ke2, 2.5.6–8])**.** *Let $(\mathcal{X}, \overline{\mathcal{X}})$ be as above and let (E, ∇) be a ∇-module on a strict neighborhood $\mathfrak{V}$ of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$. Fix a set of generators $(e_\alpha)_\alpha$ of $\Gamma(\mathfrak{V}, E)$. Then (E, ∇) is overconvergent if and only if the following condition is satisfied: For each $\eta \in (0, 1) \cap \Gamma^*$, there exists $\lambda_0 \in (0, 1) \cap \Gamma^*$ such that for any $\lambda \in [\lambda_0, 1) \cap \Gamma^*$ and any α, we have $\mathfrak{U}_\lambda \subseteq \mathfrak{V}$ and the multi-sequence*

$$\left\{ \left\| \frac{1}{i_1! \cdots i_{n+m}!} \partial_1^{i_1} \cdots \partial_{n+m}^{i_{n+m}}(e_\alpha) \right\| \eta^{i_1+\cdots+i_{n+m}} \right\}_{i_1,\ldots,i_{n+m}}$$

tends to zero as $i_1, ..., i_{n+m} \to \infty$, where $\|\cdot\|$ denotes any p-adic Banach norm on $\Gamma(\mathfrak{U}_\lambda, E)$ induced by the affinoid norm on $\Gamma(\mathfrak{U}_\lambda, \mathcal{O})$.

Remark 1.2. In [Ke2, 2.5.6], the set $\mathfrak{U}'_\lambda := \{x \in \overline{\mathcal{X}}_K \,|\, |\prod_{i=1}^n t_i(x)| \geq \lambda\}$ is used instead of $\mathfrak{U}_\lambda$. However, this does not cause any problem because we have the inclusions $\mathfrak{U}'_\lambda \subseteq \mathfrak{U}_\lambda \subseteq \mathfrak{U}'_{\lambda^n}$.

Next let us consider the case where the given formal smooth pair $(\mathcal{X}, \overline{\mathcal{X}})$ does not necessarily satisfy the condition $(*)$. Noting the fact that the functor $\Phi_{(\mathcal{X},\overline{\mathcal{X}})}$ is fully-faithful and functorial and noting the descent property of the categories $I^{\dagger}((X,\overline{X})/K)$, $\mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K)$ with respect to Zariski coverings of $\overline{\mathcal{X}}$, we see easily the following (we omit the proof):

Proposition 1.3. *Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair and let (E, ∇) be a ∇-module on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$. Then (E, ∇) is overconvergent if and only if there exists an open covering $\overline{\mathcal{X}} = \bigcup_{\alpha} \overline{\mathcal{X}}_{\alpha}$ such that $(\mathcal{X}_{\alpha}, \overline{\mathcal{X}}_{\alpha})$ (where $\mathcal{X}_{\alpha} := \overline{\mathcal{X}}_{\alpha} \cap \mathcal{X}$) satisfies the condition $(*)$ and that the restrction of (E, ∇) to a strict neighborhood of $\mathcal{X}_{\alpha,K}$ in $\overline{\mathcal{X}}_{\alpha,K}$ is overconvergent.*

By Proposition 1.3, we see that, to prove the overconvergence of (E, ∇), it suffices to check the property described in Proposition 1.1 locally on $\overline{\mathcal{X}}$.

Now we give a first step of the proof of Theorem 0.1.

Proposition 1.4. *Theorem 0.1 is true if it is true for formal smooth pairs of the form $(\widehat{\mathbb{G}}^n_{m,O_K} \times \widehat{\mathbb{A}}^m_{O_K}, \widehat{\mathbb{A}}^{n+m}_{O_K})$ $(n, m \in \mathbb{N})$.*

Proof. Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair and let (E, ∇) be a ∇-module on a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ satisfying the condition (2) in Theorem 0.1. Let $(X, \overline{X})$ be the special fiber of $(\mathcal{X}, \overline{\mathcal{X}})$. First we prove the following claim:

claim. For any closed point x of $\overline{X}$, there exist open immersions $\overline{\mathcal{U}}_x \hookrightarrow \overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$ containing x, an open sub formal scheme $\mathcal{X}_x$ of $\overline{X}_x$ and a diagram of formal smooth pairs

$$(\mathcal{X}, \overline{\mathcal{X}}) \xleftarrow{j} (\mathcal{X}_x, \overline{\mathcal{X}}_x) \xrightarrow{f} (\widehat{\mathbb{G}}^n_{m,k} \times \mathbb{A}^m_k, \widehat{\mathbb{A}}^{n+m}_k)$$

(where j is induced by the open immersion $\overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$) for some n, m such that f is a strict finite etale morphism and that the morphism $(\overline{\mathcal{U}}_x \cap \mathcal{X}_x, \overline{\mathcal{U}}_x) \longrightarrow (\overline{\mathcal{U}}_x \cap \mathcal{X}, \overline{\mathcal{U}}_x)$ induced by j is an isomorphism.

Proof of claim. Let us put $\mathcal{D} := \overline{\mathcal{X}} - \mathcal{X}$ and let D be the special fiber of $\mathcal{D}$, which is a simple normal crossing divisor in $\overline{X}$. Let $D_{\ni x} = \bigcup_{i=1}^n D_i$ be the union of irreducible components of D containing x and let $D_{\not\ni x}$ be the union of irreducible components of D not containing x. By applying [Ke1, Theorem 2] to $\overline{X} - D_{\not\ni x}$ and the simple normal crossing divisor $D - D_{\not\ni x} = D_{\ni x} - D_{\not\ni x}$ on it, we see that there exists an open subscheme $\overline{X}_x$ in $\overline{X} - D_{\not\ni x}$ containing x and a finite etale morphism $f_0 : \overline{X}_x \longrightarrow \mathbb{A}^{n+m}_k$ for some m such that, for $1 \leq i \leq n$, $f_0(D_i \cap \overline{X}_x)$ is contained in the i-th coordinate hyperplane H_i of $\mathbb{A}^{n+m}_k$. Then $D_i \cap \overline{X}_x \subseteq f_0^{-1}(H_i)$ is an open and closed immersion, and so $D \cap \overline{X}_x = D_{\ni x} \cap \overline{X}_x$ is a sub simple normal crossing divisor of $\bigcup_{i=1}^n f_0^{-1}(H_i)$ such that the complementary divisor does not contain x.

Let $f' : \overline{\mathcal{X}}'_x \longrightarrow \widehat{\mathbb{A}}^{n+m}_{O_K}$ be a finite etale morphism of formal schemes lifting f_0 and let $\mathcal{H}_i \subseteq \widehat{\mathbb{A}}^{n+m}_{O_K}$ $(1 \leq i \leq n)$ be the i-th coordinate hyperplane. Then $\bigcup_{i=1}^n f'^{-1}(\mathcal{H}_i)$

is a relative simple normal crossing divisor in $\overline{\mathcal{X}}'_x$ lifting $\bigcup_{i=1}^n f_0^{-1}(H_i)$ and so there exists a sub relative simple normal crossing divisor of $\bigcup_{i=1}^n f_0^{-1}(H_i)$ lifting $D \cap \overline{X}_x$, which we denotes by $\mathcal{D}'_x$.

On the other hand, let $\overline{\mathcal{X}}_x$ be the open sub formal scheme of $\overline{\mathcal{X}}$ lifting $\overline{X}_x$ and put $\mathcal{D}_x := \mathcal{D} \cap \overline{\mathcal{X}}_x$. Then both $(\overline{\mathcal{X}}'_x, \mathcal{D}'_x)$, $(\overline{\mathcal{X}}_x, \mathcal{D}_x)$ lifts $(\overline{X}_x, D \cap \overline{X}_x)$. Considering them as fine log formal schemes log smooth over $\mathrm{Spf}\, O_K$, we see by [Ka] that they are isomorphic. In particular, we have the isomorphism

$$\iota : (\mathcal{X} \cap \overline{\mathcal{X}}_x, \overline{\mathcal{X}}_x) = (\overline{\mathcal{X}}_x - \mathcal{D}_x, \overline{\mathcal{X}}_x) \xrightarrow{=} (\overline{\mathcal{X}}'_x - \mathcal{D}'_x, \overline{\mathcal{X}}'_x)$$

and $\mathcal{D}_x$ is a sub relative simple normal crossing divisor of $\bigcup_{i=1}^n \iota^{-1} f'^{-1}(\mathcal{H}_i)$. Let us denote the complementary divisor by $\mathcal{C}$.

Now let us put $\mathcal{X}_x := \overline{\mathcal{X}}_x - \bigcup_{i=1}^n \iota^{-1} f'^{-1}(\mathcal{H}_i)$, let f be the composite

$$\begin{aligned} (\mathcal{X}_x, \overline{\mathcal{X}}_x) &\xrightarrow{\cong} (\overline{\mathcal{X}}'_x - \bigcup_{i=1}^n f'^{-1}(\mathcal{H}_i), \overline{\mathcal{X}}'_x) \\ &\xrightarrow{f'} (\widehat{\mathbb{A}}^{n+m}_{O_K} - \bigcup_{i=1}^n \mathcal{H}_i, \widehat{\mathbb{A}}^{n+m}_{O_K}) = (\widehat{\mathbb{G}}^n_{m,O_K} \times \widehat{\mathbb{A}}^m_{O_K}, \widehat{\mathbb{A}}^{n+m}_{O_K}) \end{aligned}$$

(where the first morphism is induced by ι and the second one is induced by f') and let j be the composite

$$(\mathcal{X}_x, \overline{\mathcal{X}}_x) \longrightarrow (\overline{\mathcal{X}}_x - \mathcal{D}_x, \overline{\mathcal{X}}_x) = (\mathcal{X} \cap \overline{\mathcal{X}}_x, \overline{\mathcal{X}}_x) \xrightarrow{\subset} (\mathcal{X}, \overline{\mathcal{X}})$$

induced by the canonical inclusions. Also, let us put $\overline{\mathcal{U}}_x := \overline{\mathcal{X}}_x - \mathcal{C}$. Then we see easily that they satisfy the required properties. So the proof of the claim is finished. $\square$

Let us return to the proof of the proposition. For any closed point x in $\overline{X}$, let us choose open immersions $\overline{\mathcal{U}}_x \hookrightarrow \overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$ containing x, an open sub formal scheme $\mathcal{X}_x$ of $\overline{X}_x$ and morphisms j, f as in the above claim. Let us note that the following are equivalent:

(a) (E, ∇) is overconvergent.

(b) For any closed point x in $\overline{X}$, the restriction of (E, ∇) to $\mathrm{MIC}((\mathcal{X}_{x,K}, \overline{\mathcal{X}}_{x,K})/K)$ is overconvergent.

(c) For any closed point x in $\overline{X}$, the restriction of (E, ∇) to $\mathrm{MIC}((\mathcal{U}_{x,K} \cap \mathcal{X}_K, \overline{\mathcal{U}}_{x,K})/K) = \mathrm{MIC}((\mathcal{U}_{x,K} \cap \mathcal{X}_{x,K}, \overline{\mathcal{U}}_{x,K})/K)$ is overconvergent.

Indeed, (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) is obvious, and since each $(\mathcal{U}_x \cap \mathcal{X}_x, \overline{\mathcal{U}}_x)$ satisfies the condition $(*)$, we have (c) $\Longrightarrow$ (a) by Proposition 1.3. By this equivalence, we can replace $(\mathcal{X}, \overline{\mathcal{X}})$ by $(\mathcal{X}_x, \overline{\mathcal{X}}_x)$ to prove the overconvergence of (E, ∇). So we may assume that

there exists a strict finite etale morphism $f : (\mathcal{X}, \overline{\mathcal{X}}) \longrightarrow (\widehat{\mathbb{G}}_{m,k}^n \times \widehat{\mathbb{A}}_k^m, \widehat{\mathbb{A}}_k^{n+m})$ to prove the proposition.

In the following, we put $(\mathcal{X}_0, \overline{\mathcal{X}}_0) := (\widehat{\mathbb{G}}_{m,O_K}^n \times \widehat{\mathbb{A}}_{O_K}^m, \widehat{\mathbb{A}}_{O_K}^{n+m}), (X_0, \overline{X}_0) := (\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m, \mathbb{A}_k^{n+m})$ and let $f_K : \overline{\mathcal{X}}_K \longrightarrow \overline{\mathcal{X}}_{0,K}$ be the morphism of rigid spaces associated to f. Let $t_1, ..., t_n$ be the coordinate of $\widehat{\mathbb{G}}_{m,O_K}^n$ and let us define $\mathfrak{U}_{0,\lambda} \subseteq \overline{\mathcal{X}}_{0,K}, \mathfrak{U}_\lambda \subseteq \overline{\mathcal{X}}_K$ by

$$\mathfrak{U}_{0,\lambda} := \{x \in \overline{\mathcal{X}}_{0,K} \,|\, |t_i(x)| \geq \lambda \text{ for all } 1 \leq i \leq n\},$$
$$\mathfrak{U}_\lambda := \{x \in \overline{\mathcal{X}}_K \,|\, |t_i(x)| \geq \lambda \text{ for all } 1 \leq i \leq n\}.$$

Then f_K induces a finite etale morphism $f_K : \mathfrak{U}_\lambda \longrightarrow \mathfrak{U}_{0,\lambda}$ between affinoid rigid spaces, that is, $\Gamma(\mathfrak{U}_\lambda, \mathcal{O})$ is finite etale algebra over $\Gamma(\mathfrak{U}_{0,\lambda}, \mathcal{O})$.

For some λ, (E, ∇) is defined on $\mathfrak{U}_\lambda$. Then we can define the push-forward $f_*(E, \nabla)$ of (E, ∇) by f, which is a ∇-module on $\mathfrak{U}_{0,\lambda}$ by

$$f_*(E, \nabla) := (f_*E, f_*E \xrightarrow{f_*\nabla} f_*(E \otimes \Omega^1_{\mathfrak{U}_\lambda}) = f_*E \otimes \Omega^1_{\mathfrak{U}_{0,\lambda}}).$$

It is easy to see the following properties:

(1) For ∇-modules $(F, \nabla_F), (F', \nabla_{F'})$ on $\mathfrak{U}_\lambda$, there exists a canonical isomorphism

$$f_* : \mathrm{Hom}_{\mathfrak{U}_\lambda}((F, \nabla_F), (F', \nabla_{F'})) \xrightarrow{=} \mathrm{Hom}_{\mathfrak{U}_{0,\lambda}}(f_*(F, \nabla_F), f_*(F', \nabla_{F'})).$$

(2) For a ∇-module (F, ∇_F) on $\mathfrak{U}_{0,\lambda}$, there exists functorially the adjunction map $\mathrm{ad} : (F, \nabla_F) \longrightarrow f_*f^*(F, \nabla_F)$ and the trace map $\mathrm{tr} : f_*f^*(F, \nabla_F) \longrightarrow (F, \nabla_F)$ such that the composition $\mathrm{tr} \circ \mathrm{ad}$ is equal to the multiplication by the degree d of $\overline{\mathcal{X}}$ over $\overline{\mathcal{X}}_0$.)

By (2) above, we have the morphisms $\mathrm{ad} : f_*(E, \nabla) \longrightarrow f_*f^*f_*(E, \nabla)$, $\mathrm{tr} : f_*f^*f_*(E, \nabla) \longrightarrow f_*(E, \nabla)$ with $\mathrm{tr} \circ \mathrm{ad} = d$ and by (1), we have $\alpha : (E, \nabla) \longrightarrow f^*f_*(E, \nabla)$, $\beta : f^*f_*(E, \nabla) \longrightarrow (E, \nabla)$ with $f_*\alpha = \mathrm{ad}, f_*\beta = \mathrm{tr}$. Then we have $\beta \circ \alpha = d$ and so (E, ∇) is a direct summand of $f^*f_*(E, \nabla)$.

Now we prove that (E, ∇) is overconvergent if and only if $f_*(E, \nabla)$ is overconvergent. Indeed, if $f_*(E, \nabla)$ is overconvergent, then so is $f^*f_*(E, \nabla)$ and since (E, ∇) is a direct summand of $f^*f_*(E, \nabla)$, (E, ∇) is also overconvergent (that is, it satisfies the condition given in Proposition 1.1.) On the other hand, by [T, 5.1], the push-forward functor $f_* : I^\dagger((X, \overline{X})/K) \longrightarrow I^\dagger((X_0, \overline{X}_0)/K)$ for overconvergent isocrystals such that the diagram

$$\begin{array}{ccc} I^\dagger((X, \overline{X})/K) & \xrightarrow{\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}} & \mathrm{MIC}((\mathcal{X}_K, \overline{\mathcal{X}}_K)/K) \\ {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ I^\dagger((X_0, \overline{X}_0)/K) & \xrightarrow{\Phi_{(\mathcal{X}_0, \overline{\mathcal{X}}_0)}} & \mathrm{MIC}(((\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K})/K) \end{array}$$

is commutative, where f_* in the right vertical arrow is the functor $(F, \nabla_F) \mapsto f_*(F, \nabla_F)$ defined above. This diagram implies that $f_*(E, \nabla)$ is overconvergent if so is (E, ∇).

Now let us prove the proposition. Let (E, ∇) be as in the beginning of the proof. Let $i_0 : (\mathcal{Y}_0, \overline{\mathcal{Y}}_0) \hookrightarrow (\mathcal{X}_0, \overline{\mathcal{X}}_0)$ be any locally closed immersion of formal smooth pairs with $\dim \overline{\mathcal{Y}}_0 = 1$ and let $i : (\mathcal{Y}, \overline{\mathcal{Y}}) \hookrightarrow (\mathcal{X}, \overline{\mathcal{X}})$ be the pull-back of i_0 by f. Then $i^*(E, \nabla)$ is overconvergent by assumption, and by the argument similar to the previous paragraph (f replaced by $f|_{\overline{\mathcal{Y}}}$), $(f|_{\overline{\mathcal{Y}}})_* i^*(E, \nabla) = i_0^* f_*(E, \nabla)$ is also overconvergent. Since i_0 was arbitrary, Theorem 0.1 for $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ (which we assumed) implies the overconvergence of $f_*(E, \nabla)$ and hence (E, ∇) is also overconvergent. So we have proved the proposition. $\square$

Next we recall the definition of intrinsic generic radius of convergence of ∇-modules on polyannuli, which is due to Kedlaya-Xiao [K-X]. Let L be a field containing K complete with respect to a norm (denoted also by $|\cdot|$) which extends the given absolute value of K. A subinterval $I \subseteq [0, \infty)$ is called aligned if any endpoint of I at which it is closed is contained in Γ^* and for an aligned interval I, we define the rigid space $A_L^n(I)$ by $A_L^n(I) := \{(t_1, ..., t_n) \in \mathbb{A}_L^{n,\mathrm{an}} \,|\, \forall i, |t_i| \in I\}$.

For a formal smooth pair $(\mathcal{X}_0, \overline{\mathcal{X}}_0) := (\widehat{\mathbb{G}}_{m,O_K}^n \times \widehat{\mathbb{A}}_{O_K}^m, \widehat{\mathbb{A}}_{O_K}^m)$ and the strict neighborhood $\mathfrak{U}_{0,\lambda}$ of $\mathcal{X}_{0,K}$ $\overline{\mathcal{X}}_{0,K}$ defined in the proof of Proposition 1.4, we have $\mathcal{X}_{0,K} = A_K^n[1,1] \times A_K^m[0,1], \overline{\mathcal{X}}_{0,K} = A^{n+m}[0,1]$ and $\mathfrak{U}_{0,\lambda} = A^n[\lambda, 1] \times A^m[0,1]$. So it is important to study ∇-modules on these polyannuli.

So let L be as above, let $n, m \in \mathbb{N}$, let $\lambda \in [0,1] \cap \Gamma^*$ and let (E, ∇) be a ∇-module on $A_L^n[\lambda, 1] \times A_L^m[0,1]$. Let us denote the coordinate of $A_L^n[\lambda, 1] \times A_L^m[0,1]$ by $t_1, ..., t_{n+m}$ and put $\partial_i := \partial/\partial t_i \, (1 \leq i \leq n+m)$. For $\boldsymbol{\rho} := (\rho_i) \in [\lambda, 1]^n \times [0,1]^m$, let $L(t)_{\boldsymbol{\rho}}$ be the completion of $L(t) := L(t_1, ..., t_{n+m})$ by $\boldsymbol{\rho}$-Gauss norm. Then $L(t)_{\boldsymbol{\rho}}$ is endowed with derivations $\partial_i \, (1 \leq i \leq n+m)$ and so we can define the spectral norm of ∂_i on $L(t)_{\boldsymbol{\rho}}$, which we denote by $|\partial_i|_{\boldsymbol{\rho},L,\mathrm{sp}}$. It is easy to see that $|\partial_i|_{\boldsymbol{\rho},L,\mathrm{sp}} = p^{-1/(p-1)} \rho_i^{-1}$ in our case. On the other hand, (E, ∇) induces a differential module $E_{\boldsymbol{\rho}}$ on $L(t)_{\boldsymbol{\rho}}$ with respect to $\partial_i \, (1 \leq i \leq n+m)$ and so we can define the spectral norm of ∂_i on $E_{\boldsymbol{\rho}}$, which we denote by $|\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}}$. Then we define the intrinsic generic radius of convergence $IR(E, \boldsymbol{\rho})$ of E with radius $\boldsymbol{\rho}$ by

$$IR(E, \boldsymbol{\rho}) = \min_i \{|\partial_i|_{\boldsymbol{\rho},L,\mathrm{sp}} / |\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}}\} = \min_i \{p^{-1/(p-1)} \rho_i^{-1} |\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}}^{-1}\}.$$

Since it is known ([Ke3, 6.2.4]) that we always have $p^{-1/(p-1)} \rho_i^{-1} |\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}}^{-1} \leq 1$, $IR(E, \boldsymbol{\rho}) \leq 1$. If $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ is a basis of $E_{\boldsymbol{\rho}}$ and $G_{i,n}$ is the matrix expression of the operator ∂_i^n on $E_{\boldsymbol{\rho}}$ with respect to the basis $\mathbf{e}$, we have the equality

$$p^{-1/(p-1)} \rho_i^{-1} |\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}}^{-1} = \min\{1, p^{-1/(p-1)} \rho_i^{-1} \underline{\lim}_{n \to \infty} |G_{i,n}|_{\boldsymbol{\rho}}^{-1/n}\},$$

where $|\cdot|_{\boldsymbol{\rho}}$ denotes the maximum of the $\boldsymbol{\rho}$-Gauss norms of the entries. (See [Ke3, 6.2.5].)

The following is one of the main results of [K-X], which we use later.

Proposition 1.5 (Kedlaya-Xiao). *The function $\boldsymbol{\rho} \mapsto IR(E, \boldsymbol{\rho})$ is continuous.*

Corollary 1.6. *Fix $1 \le i \le n+m$ and for $\rho \in [0,1]$, let us denote $(1, ..., 1, \overset{i}{\check{\rho}}, 1, ..., 1)$ by $\boldsymbol{\rho}$. Then the function $\rho \mapsto p^{-1/(p-1)}\rho^{-1}|\partial_i|^{-1}_{\boldsymbol{\rho},E,\mathrm{sp}}$ is continuous.*

Proof. Let $\widetilde{L}$ be the completion of $L(t_1, ..., \check{t}_i, ..., t_{n+m})$ with respect to $(1, ..., 1)$-Gauss norm. Then (E, ∇) induces the ∇-module $(\widetilde{E}, \widetilde{\nabla})$ on $A^1_{\widetilde{L}}[\lambda, 1]$ (when $1 \le i \le n$) or on $A^1_{\widetilde{L}}[0, 1]$ (when $n+1 \le i \le n+m$), and we have the equality $|\partial_i|_{\boldsymbol{\rho},E,\mathrm{sp}} = |\partial_i|_{\rho,\widetilde{E},\mathrm{sp}}$. Hence we have $p^{-1/(p-1)}\rho^{-1}|\partial_i|^{-1}_{\boldsymbol{\rho},E,\mathrm{sp}} = R(\widetilde{E}, \rho)$ and hence it is continuous by Proposition 1.5. □

Let $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ be $((\widehat{\mathbb{G}}^n_{m,O_K} \times \widehat{\mathbb{A}}^m_{O_K}, \widehat{\mathbb{A}}^m_{O_K})$ and define the strict neighborhhood $\mathfrak{U}_{0,\lambda}$ of $\mathcal{X}_{0,K}$ in $\overline{\mathcal{X}}_{0,K}$ as before. For ∇-modules on $\mathfrak{U}_{0,\lambda,K} = A^n_K[\lambda, 1] \times A^m_K[0, 1]$, the overconvergence is described in terms of intricsic generic radius of convergence, as follows:

Proposition 1.7. *Let $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$, $\mathfrak{U}_{0,\lambda}$ be as above and let (E, ∇) be a ∇-module on $\mathfrak{U}_{0,K}$. Then the following are equivalent:*

(1) *(E, ∇) is overconvergent.*

(2) *$IR(E, \mathbf{1}) = 1$, where $\mathbf{1} := (1, ..., 1)$.*

Proof. In the proof, we will use multi-index notation as follows: For $j := (j_1, ..., j_{n+m}) \in \mathbb{N}^{n+m}$, we put $|j| := j_1 + \cdots + j_{n+m}$, $j! := j_1! \cdots j_{n+m}!$ and $\partial^j := \partial_1^{j_1} \cdots \partial_{n+m}^{j_{n+m}}$. Fix a set of generators $(e_\alpha)_\alpha$ in $\Gamma(\mathfrak{U}_{0,\lambda}, E)$. Then, by Proposition 1.1, (E, ∇) is overconvergent if and only if the following condition is satisfied: For each $\eta \in (0, 1) \cap \Gamma^*$, there exists $\rho_0 \in (\eta, 1) \cap \Gamma^*$ such that for any $\rho \in [\rho_0, 1] \cap \Gamma^*$, we have

$$\forall\alpha, \ \left\|\frac{1}{j!}\partial^j(e_\alpha)\right\| \eta^{|j|} \to 0 \ \ (j \in \mathbb{N}^{n+m}, |j| \to \infty),$$

where $\|\cdot\|$ denotes any p-adic Banach norm on $\Gamma(\mathfrak{U}_{0,\rho}, E)$ induced by the affinoid norm on $\Gamma(\mathfrak{U}_{0,\rho}, \mathcal{O})$.

First, let us note that the supremum norm on $\Gamma(\mathfrak{U}_{0,\rho}, \mathcal{O})$ gives the same topology as the affinoid norm on it by [Bo-Gu-R, 6.2.4 Theorem 1]. Hence we can replace the conclusion of the above condition by

$$\forall\alpha, \ \left|\frac{1}{j!}\partial^j(e_\alpha)\right| \eta^{|j|} \to 0 \ \ (j \in \mathbb{N}^{n+m}, |j| \to \infty),$$

where $|\cdot|$ denotes any p-adic Banach norm on $\Gamma(\mathfrak{U}_{0,\rho}, E)$ induced by the supremum norm on $\Gamma(\mathfrak{U}_{0,\rho}, \mathcal{O})$.

Next let us note that, if $\rho > \eta^{1/2}$, we have

$$\left|\frac{1}{j!}\partial^j(\sum_\alpha f_\alpha e_\alpha)\right|\eta^{|j|} \le \max_{\alpha, 0\le j'\le j}\left(\left|\frac{1}{(j-j')!}\partial^{j-j'}(f_\alpha)\right|\cdot\left|\frac{1}{j'!}\partial^{j'}(e_\alpha)\right|\eta^{|j|}\right) \tag{1.1}$$
$$\le \max_{\alpha, 0\le j'\le j}\left(|f_\alpha|\left|\frac{1}{j'!}\partial^{j'}(e_\alpha)\right|\eta^{(|j|+|j'|)/2}\right)$$

for any f_α's in $\Gamma(\mathfrak{U}_{0,\rho}, \mathcal{O})$. Using this, we see that we can replace the conclusion of the above condition by

$$\left|\frac{1}{j!}\partial^j\right|\eta^{|j|} \to 0 \quad (j \in \mathbb{N}^{n+m}, |j| \to \infty), \tag{1.2}$$

where $|\cdot|$ denotes the operator norm.

Next, note that (1.2) implies that

$$\forall i, \quad \left|\frac{1}{j!}\partial_i^j\right|\eta^j \to 0 \quad (j \in \mathbb{N}, |j| \to \infty). \tag{1.3}$$

On the other hand, if we assume (1.3), we have $\left|\frac{1}{j!}\partial_i^j\right|\eta^j \le C$ for some constant C independent of i, j. Then, for any $j = (j_1, ..., j_{n+m}) \in \mathbb{N}^{n+m}$, we have an index i with $j_i \ge |j|/(n+m)$ and the inequality

$$\left|\frac{1}{j!}\partial^j\right|\eta^{|j|} \le \prod_{l=1}^{n+m}\left|\frac{1}{j_l!}\partial_l^{j_l}\right|\eta^{j_l} \le C^{n+m-1}\left|\frac{1}{j_i!}\partial_l^{j_i}\right|\eta^{j_i}.$$

This inequality shows that (1.3) implies (1.2). Hence they are equivalent and so we can replace the conclusion of the overconvergence condition by (1.3).

Then, by using the inequality (1.1) again, we see that we can replace the conclusion of the overconvergence condition by

$$\forall\alpha, \forall i, \quad \left|\frac{1}{j!}\partial_i^j(e_\alpha)\right|\eta^j \to 0 \quad (j \in \mathbb{N}, |j| \to \infty). \tag{1.4}$$

Next, by [Ke2, 3.1.8], we have $|x| = \max_{\boldsymbol{\rho}\in\{\rho,1\}^n\times\{1\}^m}|x|_{\boldsymbol{\rho}}$ for any $x \in \Gamma(\mathfrak{U}_{0,\rho}, \mathcal{O})$, where $|\cdot|_{\boldsymbol{\rho}}$ denotes the $\boldsymbol{\rho}$-Gauss norm. Using this, we see that we can replace the conclusion of the overconvergence condition by

$$\forall\alpha, \forall i, \forall\boldsymbol{\rho} \in \{\rho, 1\}^n \times \{1\}^m, \quad \left|\frac{1}{j!}\partial_i^j(e_\alpha)\right|_{\boldsymbol{\rho},E}\eta^j \to 0 \quad (j \in \mathbb{N}, |j| \to \infty) \tag{1.5}$$

(where $|\cdot|_{\boldsymbol{\rho},E}$ denotes a norm on $E_{\boldsymbol{\rho}}$ induced by the $\boldsymbol{\rho}$-Gauss norm), and by using the analogue of the inequality (1.1), we see that it is equivalent to the condition

$$\forall i, \forall\boldsymbol{\rho} \in \{\rho, 1\}^n \times \{1\}^m, \quad \left|\frac{1}{j!}\partial_i^j\right|_{\boldsymbol{\rho}}\eta^j \to 0 \quad (j \in \mathbb{N}, |j| \to \infty) \tag{1.6}$$

(where $|\cdot|_{\boldsymbol{\rho},E}$ denotes the operator norm induced by the previous $|\cdot|_{\boldsymbol{\rho},E}$).

Then we see easily (by elementary calculus) that the overconvergence of (E,∇) is equivalent to the following assertion: For each $\eta\in(0,1)\cap\Gamma^*$, there exists $\rho_0\in(\eta,1)\cap\Gamma^*$ such that for any $\rho\in[\rho_0,1]\cap\Gamma^*$ and $\boldsymbol{\rho}=(\rho_i)_i\in\{\rho,1\}^n\times\{1\}^m$, we have

$$\min_i\{p^{-1/(p-1)}|\partial_i|^{-1}_{\boldsymbol{\rho},E,\mathrm{sp}}\}>\eta.$$

Moreover, it is easy to see that we can replace the above inequality by

$$IR(E,\boldsymbol{\rho})=\min_i\{p^{-1/(p-1)}\rho_i^{-1}|\partial_i|^{-1}_{\boldsymbol{\rho},E,\mathrm{sp}}\}>\eta.$$

Then, using the contunuity of the function $\boldsymbol{\rho}\mapsto IR(E,\boldsymbol{\rho})$ (Proposition 1.5), we can conclude that the overconvergence condition for (E,∇) is equivalent to the condition $IR(E,\mathbf{1})=1$. □

Before the proof of the main theorem, finally we recall a technical lemma, which was proved in [S, 2.8].

Lemma 1.8. *Let $\mathcal{X}$ be a smooth p-adic formal scheme over $\operatorname{Spf} O_K$, let $I\subseteq(0,1)$ be a closed aligned interval of positive length and let $a=\sum_{n\in\mathbb{Z}}a_nt^n$ be a non-zero element of $\Gamma(\mathcal{X}_K\times A^1_K(I),\mathcal{O})$. Then there exist an open dense sub affine formal scheme $\mathcal{U}\subseteq\mathcal{X}$ and a closed aligned subinterval $I'\subseteq I$ of positive length satisfying the following conditions*:

(1) *$a\in\Gamma(\mathcal{U}_K\times A^1_K(I'),\mathcal{O}^\times)$.*

(2) *For any $u\in\mathcal{U}_K$ and $\rho\in I'$, we have $|a(u)|_\rho=|a|_\rho$, where $a(u):=\sum_{n\in\mathbb{Z}}a_n(u)t^n\in\Gamma(u\times A^1_K(I'),\mathcal{O})$ and $|\cdot|_\rho$ denotes the ρ-Gauss norm.*

Proof. For the convenience of the reader, we reproduce the proof here. In this proof, $|\cdot|$ denotes the supremum norm. Let us write $I=[\alpha,\beta]$. By [Ke2, 3.1.7, 3.1.8], we have

$$|a|=\max(\sup_n(|a_n|\alpha^n),\sup_n(|a_n|\beta^n))=\max(\sup_{n\le 0}(|a_n|\alpha^n),\sup_{n\ge 0}(|a_n|\beta^n)).$$

Let us define finite subsets $A\subseteq\mathbb{Z}_{\le 0}$, $B\subseteq\mathbb{Z}_{\ge 0}$ by $A:=\{n\le 0\,|\,|a_n|\alpha^n=|a|\}$, $B:=\{n\ge 0\,|\,|a_n|\beta^n=|a|\}$. Then we have $A\cup B\neq\emptyset$.

Let us first consider the case $A\neq\emptyset$. Let n_0 be the maximal element of A. Then, since $a_{n_0}\neq 0$, there exists an element $b\in K^\times$ such that $ba_{n_0}\in\Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})$ and that the image $\overline{ba_{n_0}}$ of ba_{n_0} in $\Gamma(X,\mathcal{O}_X)$ is non-zero. Let $\mathcal{U}\subseteq\mathcal{X}$ be the open dense affine sub formal scheme such that $\overline{ba_{n_0}}$ is invertible on $\mathcal{U}\times_\mathcal{X}X$. Then we have $ba_{n_0}\in\Gamma(\mathcal{U},\mathcal{O}_\mathcal{U}^\times)$. So, for all $u\in\mathcal{U}_K$, we have $|a_{n_0}(u)|=|b^{-1}|$ and hence $|a_{n_0}(u)|=|a_{n_0}|$. (Here note that, for elements in $\Gamma(\mathcal{X}_K,\mathcal{O})$, its supremum norm on $\mathcal{X}_K$ is the same as that on $\mathcal{U}_K$.) Next we prove the following claim:

claim. There exists a closed aligned subinterval $I' \subseteq I$ of positive length such that $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$ for any $n \in \mathbb{Z}, \neq n_0$ and $\rho \in I'$.

Let us put $C := \{n \in \mathbb{Z} \mid \max(|a_n|\alpha^n, |a_n|\beta^n) \geq |a_{n_0}|\beta^{n_0}\}$. Then C is a finite set containing A. If $n \in A, \neq n_0$, we have $|a_n|\alpha^n = |a_{n_0}|\alpha^{n_0}$ and $n < n_0 \leq 0$. Hence we have $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$ for any $\rho \in (\alpha, \beta]$. For $n \in C - A$, we have $|a_n|\alpha^n < |a_{n_0}|\alpha^{n_0}$. So there exists $\beta' \in (\alpha, \beta]$ such that, for any $n \in C - A$ and for any $\rho \in [\alpha, \beta']$, we have $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$. For $n \notin C$, we have, for any $\rho \in I$, the inequalities

$$|a_n|\rho^n \leq \max(|a_n|\alpha^n, |a_n|\beta^n) < |a_{n_0}|\beta^{n_0} \leq |a_{n_0}|\rho^{n_0}.$$

Summing up these, we see the claim.

Let us put $f := \sum_{n\neq n_0}(a_n/a_{n_0})t^{n-n_0} \in \Gamma(\mathcal{U}_K \times A^1_K(I'), \mathcal{O})$ and take any $u \in \mathcal{U}_K$, $\rho \in I'$. Then we have

$$|f(u)|_\rho \leq \frac{\sup_{n\neq n_0}(|a_n|\rho^n)}{|a_{n_0}(u)|\rho^{n_0}} = \frac{\sup_{n\neq n_0}(|a_n|\rho^n)}{|a_{n_0}|\rho^{n_0}} < 1.$$

So we have $|f| < 1$. So we have $a = a_{n_0}t^{n_0}(1+f) \in \Gamma(\mathcal{U}_K \times A^1_K(I'), \mathcal{O}^\times)$. Moreover, for any $u \in \mathcal{U}_K$ and $\rho \in I'$, we have

$$|a(u)|_\rho = \sup_n(|a_n(u)|\rho^n) = |a_{n_0}(u)|\rho^{n_0} = |a_{n_0}|\rho^{n_0} = \sup_n(|a_n|\rho^n) = |a|_\rho.$$

We can prove the lemma in the case $B \neq \emptyset$ in the same way. (In this case, we define n_0 to be the minimal element of B.) So we are done. □

Now we are ready to prove Theorem 0.1. The proof is similar to that of [S, 2.5, 2.9].

Proof of Theorem 0.1. As we explanined in the introduction, it suffices to prove the implication (2) $\Longrightarrow$ (1) and by Proposition 1.4, it suffices to prove it for a formal smooth pair of the form $(\widehat{\mathbb{G}}^n_{m,O_K} \times \widehat{\mathbb{A}}^m_{O_K}, \widehat{\mathbb{A}}^{n+m}_{O_K}) =: (\mathcal{X}_0, \overline{\mathcal{X}}_0)$. (Then we have $\mathcal{X}_{0,K} = A^n_K[1,1] \times A^m_K[0,1], \overline{\mathcal{X}}_{0,K} = A^{n+m}_K[0,1]$.) For $\lambda \in (0,1] \cap \Gamma^*$, let $\mathfrak{U}_{0,\lambda} = A^n_K[\lambda,1] \times A^m_K[0,1]$ be as before.

Let (E, ∇) be a ∇-module on a strict neighborhood of $\mathcal{X}_{0,K}$ in $\overline{\mathcal{X}}_{0,K}$ satisfying the condition (2). Then it is defined on $\mathfrak{U}_{0,\lambda}$ for some λ. Let us assume that it is not overconvergent. Then, by Proposition 1.7, we have $IR(E, \mathbf{1}) < 1$. So we have $p^{-1/(p-1)}|\partial_i|^{-1}_{\mathbf{1},E,\mathrm{sp}} < 1$ for some $i\,(1 \leq i \leq n+m)$. Fix such index i.

In the following, for $\rho \in [0,1]$, we denote $(1, ..., 1, \overset{i}{\check{\rho}}, 1, ..., 1) \in [0,1]^{n+m}$ simply by $\boldsymbol{\rho}$. Let us denote the restriction of E to

$$\begin{aligned}(\widehat{\mathbb{G}}^{n+m-1}_{m,O_K})_K \times A^1_K[\lambda,1] &= A^{n+m-1}_K[1,1] \times A^1_K[\lambda,1] \\ &\cong A^{i-1}_K[1,1] \times A^1_K[\lambda,1] \times A^{n+m-i}_K[1,1] \quad (\text{if } 1 \leq i \leq n), \\ (\widehat{\mathbb{G}}^{n+m-1}_{m,O_K})_K \times A^1_K[0,1] &= A^{n+m-1}_K[1,1] \times A^1_K[0,1] \\ &\cong A^{i-1}_K[1,1] \times A^1_K[0,1] \times A^{n+m-i}_K[1,1] \quad (\text{if } n+1 \leq i \leq n+m)\end{aligned}$$

(where the second isomorphism is the permutation of the i-th factor and the last factor) also by E.

Let us take a closed aligned intervals $I_r = [\alpha_r, \beta_r] \subseteq [\lambda, 1)\,(r \in \mathbb{N})$ with $\alpha_r < \beta_r < \alpha_{r+1}\,(\forall r)$, $\lim_{r\to\infty} \alpha_r = 1$ and put $A_r := \Gamma((\widehat{\mathbb{G}}^{n+m-1}_{m,O_K})_K \times A_K(I), \mathcal{O})$, $\mathbf{E}_r := \Gamma((\widehat{\mathbb{G}}^{n+m-1}_{m,O_K})_K \times A_K(I_r), E)$. Then A_r is an integral domain and $\mathbf{E}_r$ is a finitely generated A-module. Let $\mathbf{e}_r := (\mathbf{e}_{r,1}, ..., \mathbf{e}_{r,\mu})$ be a basis of $\operatorname{Frac} A_r \otimes_{A_r} \mathbf{E}_r$ as $\operatorname{Frac} A_r$-vector space ($\mu = \operatorname{rk} E$) and let $(\mathbf{f}_{r,1}, ..., \mathbf{f}_{r,\nu_r})$ be a set of generator of $\mathbf{E}_r$ as A_r-module. Then there exist $b_{r,ij} := b'_{r,ij}/b''_{r,ij}, c_{r,ji} := c'_{r,ji}/c''_{r,ji} \in \operatorname{Frac} A_r\,(1 \leq i \leq \mu, 1 \leq j \leq \nu_r)$ such that $\mathbf{e}_{r,i} = \sum_{j=1}^{\nu_r} b_{r,ij}\mathbf{f}_{r,j}\,(\forall i), \mathbf{f}_{r,j} = \sum_{i=1}^{\mu} c_{r,ji}\mathbf{e}_{r,i}\,(\forall j)$. By Lemma 1.8, there exists an open dense sub affine formal scheme $\mathcal{V}_r \subseteq (\widehat{\mathbb{G}}^{n+m-1}_{m,O_K})_K$ and closed aligned subinterval $I'_r \subseteq I_r$ of positive length such that $b''_{r,ij}, c''_{r,ji} \in \Gamma(\mathcal{V}_K \times A^1_K(I'), \mathcal{O}^\times)$. Then we see that $\mathbf{e}_r$ forms a basis of $\Gamma(\mathcal{V}_{r,K} \times A^1_K(I'_r), E)$ as $\Gamma(\mathcal{V}_{r,K} \times A^1_K(I'_r), \mathcal{O})$-module. For $s \in \mathbb{N}$, let $G_{r,s} \in \operatorname{Mat}_\mu(\Gamma(\mathcal{V}_{r,K} \times A^1_K(I'_r), \mathcal{O}))$ be the matrix expression of ∂^s with respect to the basis $\mathbf{e}_r$ and let us put $V_r := \mathcal{V}_r \times_{\mathcal{Z}} Z$. Then, again by Lemma 1.8, there exists a decreasing sequence $\{V_{r,s}\}_{s\in\mathbb{N}}$ of dense open subschemes in V_r and a decreasing sequence $\{I'_{r,s}\}_{s\in\mathbb{N}}$ of closed aligned subintervals of I'_r of positive length such that, for any separable closed point x in $V_{r,s}$, any lift $\widetilde{x}$ of x in $\widehat{\mathbb{G}}^{n+m-1}_{m,O_K}$, any $s' \leq s$ and any $\rho_r \in I'_{r,s}$, we have the equality $|G_{r,s'}(\widetilde{x}_K)|_{\rho_r} = |G_{r,s'}|_{\boldsymbol{\rho}_r}$, where $G_{r,s'}(\widetilde{x}_K) \in \operatorname{Mat}_\mu(\Gamma(\widetilde{x}_K \times A^1_K(I'_r), \mathcal{O}))$ is the pull-back of $G_{r,s'}$ by $\widetilde{x}_K \times A^1_K(I'_r) \hookrightarrow \mathcal{V}_{r,K} \times A^1_K(I'_r)$. Let $\partial_i(\widetilde{x}_K)$ be the action of ∂_i on the pull-back $E(\widetilde{x}_K)$ of E to $\widetilde{x}_K \times A^1_K[\lambda, 1]$ (if $1 \leq i \leq n$), $\widetilde{x}_K \times A^1_K[0, 1]$ (if $n+1 \leq i \leq n+m$). Then, for any separable closed point x in $\bigcap_s V_{r,s}$, any lift $\widetilde{x}$ of x in $\widehat{\mathbb{G}}^{n+m-1}_{m,O_K}$ and any $\rho_r \in \bigcap_{s\in\mathbb{N}} I'_{r,s}$, we have

$$\begin{aligned}(1.7)\qquad p^{-1/(p-1)}\rho_r^{-1}|\partial_i(\widetilde{x}_K)|^{-1}_{\rho_r,E(\widetilde{x}_K),\mathrm{sp}} &= \min\{1, p^{-1/(p-1)}\rho_r^{-1} \varliminf_{s\to\infty} |G_{r,s}(\widetilde{x}_K)|^{-1/s}_{\rho_r}\} \\ &= \min\{1, p^{-1/(p-1)}\rho_r^{-1} \varliminf_{s\to\infty} |G_{r,s}|^{-1/s}_{\boldsymbol{\rho}_r}\} \\ &= p^{-1/(p-1)}\rho_r^{-1}|\partial_i|^{-1}_{\boldsymbol{\rho}_r,E,\mathrm{sp}}.\end{aligned}$$

Now let us take a separable point x in $\bigcap_{r,s} V_{r,s}$ (which is possible since k is uncountable) and its lift $\widetilde{x}$ in $\widehat{\mathbb{G}}^{n+m-1}_{m,O_K}$. Then we have the equality (1.7) for any $r \in \mathbb{N}$. Since the functions

$$\rho \mapsto p^{-1/(p-1)}\rho^{-1}|\partial_i(\widetilde{x}_K)|^{-1}_{\rho,E(\widetilde{x}_K),\mathrm{sp}}, \quad \rho \mapsto p^{-1/(p-1)}\rho^{-1}|\partial_i|^{-1}_{\boldsymbol{\rho}_r,E,\mathrm{sp}}.$$

are continuous by Proposition 1.5 and Corollary 1.6, this implies

$$(1.8)\qquad IR(E(\widetilde{x}_K), 1) = p^{-1/(p-1)}|\partial_i|^{-1}_{\mathbf{1},E,\mathrm{sp}} < 1.$$

Now let us define the strict closed immersion $i : (\mathcal{Y}, \overline{\mathcal{Y}}) \hookrightarrow (\mathcal{X}_0, \overline{\mathcal{X}}_0)$ of formal smooth pairs by $\overline{\mathcal{Y}} := \widetilde{x} \times \widehat{\mathbb{A}}^1_{O_K}$,

$$\begin{aligned} i : \overline{\mathcal{Y}} = \widetilde{x} \times \widehat{\mathbb{A}}^1_{O_K} &\hookrightarrow \widehat{\mathbb{G}}^{n+m-1}_{O_K} \times \widehat{\mathbb{A}}^1_{O_K} \\ &\cong \widehat{\mathbb{G}}^{i-1}_{O_K} \times \widehat{\mathbb{A}}^1_{O_K} \times \widehat{\mathbb{G}}^{n+m-i}_{O_K} \hookrightarrow \widehat{\mathbb{A}}^{n+m}_{O_K} = \overline{\mathcal{X}}_0 \end{aligned}$$

(where the isomorphism in the second line is the permutation of the i-th factor and the last factor) and

$$\mathcal{Y} := \overline{\mathcal{Y}} \times_{\overline{\mathcal{X}}_0} \mathcal{X}_0 = \begin{cases} \widetilde{x} \times \widehat{\mathbb{G}}_{m,O_K}, & \text{if } 1 \le i \le n, \\ \widetilde{x} \times \widehat{\mathbb{A}}^1_{O_K}, & \text{if } n+1 \le i \le n+m. \end{cases}$$

Then $E(\widetilde{x}_K)$ is nothing but the pull-back i^*E of E by i. Hence, by (1.8) and Proposition 1.7, $i^*(E,\nabla)$ is not overconvergent and this contradicts the assumption on (E,∇). Hence (E,∇) is overconvergent and so the proof of the theorem is finished. $\square$

Remark 1.9. Let $(\mathcal{X},\overline{\mathcal{X}})$ be a formal smooth pair admitting a strict finite etale morphism $f : (\mathcal{X},\overline{\mathcal{X}}) \longrightarrow (\mathcal{X}_0,\overline{\mathcal{X}}_0) := (\widehat{\mathbb{G}}^n_{m,O_K} \times \widehat{\mathbb{A}}^m_{O_K}, \widehat{\mathbb{A}}^{n+m}_{O_K})$. For a closed sub formal scheme $\widetilde{x} \hookrightarrow \widehat{\mathbb{G}}^{n+m-1}_{m,O_K}$ which is etale over $\mathrm{Spf}\, O_K$ and $1 \le i \le n+m$, let $\overline{\mathcal{Y}}_{\widetilde{x},i,0} \hookrightarrow \overline{\mathcal{X}}_0$ be the closed immersion defined by

$$\begin{aligned} \overline{\mathcal{Y}}_{\widetilde{x},i,0} &:= \widetilde{x} \times \widehat{\mathbb{A}}^1_{O_K} \hookrightarrow \widehat{\mathbb{G}}^{n+m-1}_{O_K} \times \widehat{\mathbb{A}}^1_{O_K} \\ &\cong \widehat{\mathbb{G}}^{i-1}_{O_K} \times \widehat{\mathbb{A}}^1_{O_K} \times \widehat{\mathbb{G}}^{n+m-i}_{O_K} \hookrightarrow \widehat{\mathbb{A}}^{n+m}_{O_K} = \overline{\mathcal{X}}_0 \end{aligned}$$

(where the isomorphism in the second line is the permutation of the i-th factor and the last factor), let $\mathcal{Y}_{\widetilde{x},i,0} := \overline{\mathcal{Y}}_{\widetilde{x},i,0} \times_{\overline{\mathcal{X}}_0} \mathcal{X}_0$ and let $(\mathcal{Y}_{\widetilde{x},0},\overline{\mathcal{Y}}_{\widetilde{x},0})$ be the pull-back of $(\mathcal{Y}_{\widetilde{x},0},\overline{\mathcal{Y}}_{\widetilde{x},0})$ by f. Then, by the proofs of Proposition 1.4 and Theorem 0.1 given above, we see the following refined version of Theorem 0.1 in this case: For $(E,\nabla) \in \mathrm{MIC}((\mathcal{X}_K,\overline{\mathcal{X}}_K)/K)$, the following are equivalent:

(1) (E,∇) is overconvergent.

(2) For any $\widetilde{x}$ as above and for any $1 \le i \le n+m$, the restriction of (E,∇) to $\mathrm{MIC}((\mathcal{Y}_{\widetilde{x},i,K},\overline{\mathcal{Y}}_{\widetilde{x},i,K})/K)$ is overconvergent.

2 Proof of the second main theorem

In this section, we give a proof of the second main theorem (Theorem 0.2) in this paper. To do so, first we prove a partial generalization of a result of Kedlaya [Ke1, Theorem 2] on etale covers of smooth k-varieties to the case of smooth formal schemes and smooth schemes over O_K. Throughout this section, let π be a fixed uniformizer of O_K and for a scheme $\mathbf{X}$ over $\mathrm{Spec}\, O_K$, we denote the generic fiber of it by $\mathbf{X}_K$, the special fiber of it by $\mathbf{X}_k$ and the p-adic completion of it by $\widehat{\mathbf{X}}$. For a p-adic formal scheme $\mathcal{X}$ over $\mathrm{Spf}\, O_K$, we denote the special fiber of it by $\mathcal{X}_k$.

Let $(\mathcal{X},\overline{\mathcal{X}})$ be a formal smooth pair and let x be a closed point of $\mathcal{X}_k$. Let us put $\mathcal{D} := \overline{\mathcal{X}} - \mathcal{X} = \bigcup_{i=1}^q \mathcal{D}_i$ (where $\mathcal{D}_i$'s are connected and smooth over $\mathrm{Spf}\, O_K$)

and define $I_{\ni x}, I_{\not\ni x}, \mathcal{D}_{\ni x}, \mathcal{D}_{\not\ni x}$ in the following way:

$$I_{\ni x} := \{i \mid x \in \mathcal{D}_i\}, \qquad I_{\not\ni x} := \{i \mid x \notin \mathcal{D}_i\},$$
$$\mathcal{D}_{\ni x} := \bigcup_{i \in I_{\ni x}} \mathcal{D}_i, \qquad \mathcal{D}_{\not\ni x} := \bigcup_{i \in I_{\not\ni x}} \mathcal{D}_i.$$

We say that the formal smooth pair $(\mathcal{X}, \overline{\mathcal{X}})$ satisfies the condition $(*)_x$ if, for any $I \subseteq I_{\ni x}$, $\bigcap_{i \in I} \mathcal{D}_i$ is irreducible. We will prove the following result, which is a formal scheme version of [Ke1, Theorem 2]:

Theorem 2.1. *Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair with $\dim \overline{\mathcal{X}} = n$ and $\overline{\mathcal{X}}$ projective over $\operatorname{Spf} O_K$. Let x be a closed point in $\overline{\mathcal{X}}_k$ and assume that $(\mathcal{X}, \overline{\mathcal{X}})$ satisfies the condition $(*)_x$. Let us put $\mathcal{D} := \overline{\mathcal{X}} - \mathcal{X}$, let us define $\mathcal{D}_{\ni x}, \mathcal{D}_{\not\ni x}$ as above and let $\mathcal{D}_{\ni x} = \bigcup_{i=1}^r \mathcal{D}_i$ be the decomposition of $\mathcal{D}_{\ni x}$ into irreducible components. Then there exists a finite flat morphism $f : \overline{\mathcal{X}} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$ satisfying the following conditions*:

(1) *f is etale on $f^{-1}(\widehat{\mathbb{A}}^n_{O_K})$.*

(2) *We have $\mathcal{D}_{\not\ni x} \subseteq f^{-1}(H_0)$ and $x \in f^{-1}(\widehat{\mathbb{A}}^n_{O_K})$, where $H_0 := \widehat{\mathbb{P}}^n_{O_K} - \widehat{\mathbb{A}}^n_{O_K} \subseteq \widehat{\mathbb{P}}^n_{O_K}$ is the hyperplane at infinity.*

(3) *For $1 \leq i \leq r$, we have $\mathcal{D}_i \subseteq f^{-1}(H_i)$, where $H_i \subseteq \widehat{\mathbb{P}}^n_{O_K}$ is the i-th coordinate hyperplane. (Hence, by combining with (1), we see that $\mathcal{D}_i \cap f^{-1}(\widehat{\mathbb{A}}^n_{O_K}) \subseteq f^{-1}(H_i \cap \widehat{\mathbb{A}}^n_{O_K})$ is an open and closed immersion.)*

We prove Theorem 2.1 by mimicking the proof of [Ke1, Theorem 2]. First we prove several premilinary lemmas.

Lemma 2.2 (cf. [Ke1, Lemma 3])**.** *Let $\mathcal{X}$ be a p-adic formal scheme projective over $\operatorname{Spf} O_K$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$. Let $\alpha_0, \ldots, \alpha_n$ be sections of $\mathcal{L}$ such that the intersection of zero loci of α_i $(0 \leq i \leq n)$ is empty. Then α_i's induce a finite morphism $\alpha : \mathcal{X} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$.*

Proof. Since α is proper, it suffices to check that α modulo πO_K is quasi-finite and it is proved in [Ke1, Lemma 3]. □

Lemma 2.3 (cf. [Ke1, Lemma 4])**.** *Let $\mathcal{X}$ be a p-adic formal scheme projective and flat over $\operatorname{Spf} O_K$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$. Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed sub formal scheme and let $\mathcal{Z} \subseteq \mathcal{X}$ be a 0-dimensional closed sub formal scheme such that $\mathcal{D} \cap \mathcal{Z}$ is empty. Then, for l sufficiently divisible, there exists a section of $\mathcal{L}^{\times l}$ which vanishes on $\mathcal{D}$ but not on any point in $\mathcal{Z}_k$.*

Proof. Let $\mathbf{X}, \mathbf{L}, \mathbf{D}$ be the algebraizations of $\mathcal{X}, \mathcal{L}, \mathcal{D}$, respectively. Then, for sufficiently large $a > 0$, the map $\Gamma(\mathbf{X}, \mathbf{L}^{\otimes a}) = \Gamma(\mathcal{X}, \mathcal{L}^{\otimes a}) \longrightarrow \Gamma(\mathcal{X}_k, \mathcal{L}^{\otimes a})$ is surjective. By this fact and the proof of [Ke1, Lemma 4], we see that, for a sufficiently large,

there exists a section s_0 of $\mathbf{L}^{\otimes a}$ which does not vanish on any point in $\mathcal{Z}_k$. On the complement of the zero locus of s_0 in $\mathbf{X}$, there exists a regular function vanishing on $\mathbf{D}$, not on any point in $\mathcal{Z}_k$ and this function has the form $s_1 s_0^{-b}$ for some b. Then, for any l divisible by ab, the image of the section $s_1^{l/ab} \in \Gamma(\mathbf{X}, \mathbf{L}^{\otimes l})$ in $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes l})$ has the desired property. ◻

Lemma 2.4 (cf. [Ke1, Lemma 5]). *Let $\mathcal{X}$ be a p-adic formal scheme projective and flat over $\operatorname{Spf} O_K$ with $\dim \mathcal{X} = n$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$. Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed sub formal scheme and let $\mathcal{Z} \subseteq \mathcal{X}$ be a 0-dimensional closed sub formal scheme such that $\mathcal{D} \cap \mathcal{Z}$ is empty. Let m be an integer with $0 \leq m \leq n$ and let $\mathcal{D}_1, ..., \mathcal{D}_m$ be divisors of $\mathcal{X}$ such that, for any $T \subseteq [1, m]$, $\mathcal{D} \cap \bigcap_{t \in T} \mathcal{D}_t$ has codimension at least $|T|$ in $\mathcal{D}$. Then, for l sufficiently divisible, there exist sections $s_1, ..., s_n$ of $\mathcal{L}^{\otimes l}$ with no common zero on $\mathcal{D}_k$ such that s_i vanishes on $\mathcal{Z}$ $(1 \leq i \leq n)$ and that s_i vanishes on $\mathcal{D}_i$ for $1 \leq i \leq m$.*

Proof. Although the following proof is the same as that in [Ke1, Lemma 5], we reproduce the proof here for the reader's convenience. We construct the desired section inductively. Suppose that, for $0 \leq j < n$, there exist positive integers l_i and sections s'_i of $\mathcal{L}^{\otimes l_i}$ $(1 \leq i \leq j)$ satisfying the following conditions:

(a) Each s'_i vanishes on $\mathcal{Z}$.

(b) For $1 \leq i \leq \min(j, m)$, s'_i vanishes on $\mathcal{D}_i$.

(c) For any $T \subseteq [j+1, m]$, $Y_{j,T} := \mathcal{D} \cap \bigcap_{t \in T} \mathcal{D}_t \cap \bigcap_{i=1}^{j} \{\text{zero locus of } s'_i\}$ has codimension $\geq j + |T|$ in $\mathcal{D}$.

(This is true for $j = 0$ by hypothesis.) Then let $\mathcal{Z}_j$ be a 0-dimensional closed sub formal scheme of $\mathcal{D} - \mathcal{D}_{j+1}$ which meets each irreducible component of $Y_{j,T}$ having codimension $j + |T|$ in $\mathcal{D}$, for each $T \subseteq [j+2, m]$. (By (c), none of these components are contained in $\mathcal{D} \cap \mathcal{D}_{j+1}$. Hence such $\mathcal{Z}_j$ actually exists.) By Lemma 2.3, for l_{j+1} sufficiently divisible, there exists a section s'_{j+1} of $\mathcal{L}^{\otimes l_{j+1}}$ vanishing along $\mathcal{Z} \cup \mathcal{D}_{j+1}$, not on $(\mathcal{Z}_j)_k$. Then, with this s'_{j+1}, the conditions (a), (b), (c) are satisfied with j replaced by $j + 1$.

So we can take sections $s'_1, ..., s'_n$ such that the conditions (a), (b), (c) are satisfied with $j = n$. Let l_0 be the least common multiple of l_j's. Then for any l divisible by l_0, the sections $s_i := s_i'^{l/l_i}$ $(1 \leq i \leq n)$ of $\mathcal{L}^{\otimes l}$ satisfy the desired properties. ◻

Lemma 2.5 (cf. [Ke1, Lemma 6]). *Let $\mathcal{X}$ be a p-adic formal scheme projective and smooth over $\operatorname{Spf} O_K$ with $\dim \mathcal{X} = n$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$. Let $\bigcup_{i=1}^{m} \mathcal{D}_i$ be a relative simple normal crossing divisor in $\mathcal{X}$ (each $\mathcal{D}_i$ being smooth) and let $\widetilde{x}$ be a closed sub formal scheme of $\bigcap_{i=1}^{m} \mathcal{D}_i$ connected and etale over $\operatorname{Spf} O_K$. Let α be a section of $\mathcal{L}$ whose zero locus $\mathcal{D}$ does not meet $\widetilde{x}$. Then, for l sufficiently divisible, there exist sections $\delta_1, ..., \delta_n$ of $\mathcal{L}^{\otimes l}$ satisfying the following conditions*:

(1) *δ_i's have no common zero on $\mathcal{D}$, that is, $\bigcap_{i=1}^n\{$zero locus of $\delta_i\} \cap \mathcal{D}$ is empty. Hence $\alpha^l, \delta_1, ..., \delta_n$ define a finite morphism $f : \mathcal{X} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$ by Lemma 2.2.*

(2) *f is unramified at $\widetilde{x}_k$.*

(3) *For $1 \leq i \leq m$, δ_i vanishes on $\mathcal{D}_i$.*

Proof. Let $\mathbf{X}, \bigcup_{i=1}^m \mathbf{D}_i, \mathbf{L}, \mathbf{D}$ be the algebraization of $\mathcal{X}, \bigcup_{i=1}^m \mathcal{D}_i, \mathcal{L}, \mathcal{D}$. Then α induces a section $\widetilde{\alpha}$ of $\mathbf{L}$ with zero locus $\mathbf{D}$. Hence $\mathbf{U} := \mathbf{X} - \mathbf{D}$ is an affine scheme. Then we can find $\widetilde{f_1}, ..., \widetilde{f_n} \in \Gamma(\mathbf{U}, \mathcal{O})$ such that $\widetilde{f_i}$ vanishes on $\mathbf{D}_i$ for $1 \leq i \leq m$ and that they induces a morphism $\mathbf{U} \longrightarrow \mathbb{A}^n_{O_K}$ which is unramified at $\widetilde{x}_k \in \mathcal{X}_k = \mathbf{X}_k \subseteq \mathbf{X}$. Write $\widetilde{f_i} = \widetilde{\beta_i}\widetilde{\alpha}^{-j_i}$ for some positive integer j_i and $\beta_i \in \Gamma(\mathbf{X}, \mathbf{L}^{\otimes j_i})$.

Put $\mathcal{U} := \mathcal{X} - \mathcal{D}$ and denote the image of $\widetilde{f_i}$ (resp. $\widetilde{\beta_i}$) in $\Gamma(\mathcal{U}, \mathcal{O})$ (resp. $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes j_i})$ by f_i (resp. β_i). Then f_i's induce a map $\mathcal{U} \longrightarrow \widehat{\mathbb{A}}^n_{O_K}$ which is unramified at $\widetilde{x}_k$. By Lemma 2.4, for l sufficiently divisible, we can choose $\gamma_1, ..., \gamma_n \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes l})$ such that $\gamma_i\,(1 \leq i \leq n)$ vanishes on $\widetilde{x}$, γ_i vanishes on $\mathcal{D}_i$ for $1 \leq i \leq m$ and that $\gamma_1, ..., \gamma_m$ has no common zero on $\mathcal{D}$. Now let us put $\delta_i := \beta_i\alpha^{2l-j_i} + \gamma_i^2 \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes 2l})$ for $1 \leq i \leq n$. Then we can check that this δ_i's satisfy the required properties. □

Now we give a proof of Theorem 2.1:

Proof of Theorem 2.1. The proof is essentially the same as that of [Ke1, Theorem 2]. Let $\mathcal{L}$ be an ample line bundle on $\overline{\mathcal{X}}$ and let $\widetilde{x}$ be a lift of x in $\mathcal{D}_{\ni x}$. Since we may replace $\mathcal{L}$ by its tensor power, we may assume by Lemma 2.3 that there exists a section s of $\mathcal{L}$ which vanishes on $\mathcal{D}_{\not\ni x}$ and not on x. By Lemma 2.5, there exist a positive integer m and sections $s_1, ..., s_n$ of $\mathcal{L}^{\otimes m}$ satisfying the following conditions:

(a) s_i's have no common zero on $\mathcal{D}_{\not\ni x}$ and so $s^m, s_1, ..., s_n$ defines a finite morphism $g : \overline{\mathcal{X}} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$ by Lemma 2.2.

(b) g is unramified at x.

(c) s_i vanishes on $\mathcal{D}_i$ for $1 \leq i \leq r$.

Let $\mathcal{D}$ be the zero locus of s. (Then we have $\mathcal{D}_{\not\ni x} \subseteq \mathcal{D}$ and $\mathcal{D}$ does not contain x.) The locus on $\overline{\mathcal{X}} - \mathcal{D}$ where g is unramified is open. Let $\mathcal{E}$ be its complement in $\mathcal{X}$ and endow $\mathcal{E}$ a structure of a closed sub formal scheme of $\mathcal{X}$ satisfying $\mathcal{D} \subseteq \mathcal{E}$. (Note that we have $x \notin \mathcal{E}$.) Then, by Lemma 2.3, there exists some q and $t \in \Gamma(\overline{\mathcal{X}}, \mathcal{L}^{\otimes qm})$ vanising on $\mathcal{E}$ and not on x. Let $\mathcal{Z}$ be the vanishing locus of t (so $\mathcal{E} \subseteq \mathcal{Z}, x \notin \mathcal{Z}$ and $\mathcal{Z}$ is pure codimension 1 in $\overline{\mathcal{X}}$). For any $T \subseteq [1, r]$, $\bigcap_{t \in T} \mathcal{D}_t$ is irreducible by the assumption $(*)_x$ and it contains x. Hence $\mathcal{Z} \cap \bigcap_{t \in T} \mathcal{D}_t$ has codimension $|T|$ in $\mathcal{Z}$. So we can apply Lemma 2.4 and we can conclude that there exist some l and

sections $t_1, ..., t_n \in \Gamma(\overline{\mathcal{X}}, \mathcal{L}^{\otimes lqm})$ with no common zero on $\mathcal{Z}$ such that t_i vanishes on $\mathcal{D}_i$ for $1 \leq i \leq r$. Now let us put

$$\begin{aligned} u_0 &:= t^{pl}, \\ u_i &:= s_i s^{m(pq-1)} t^{p(l-1)} + t_i^p \quad (1 \leq i \leq n). \end{aligned}$$

Then they defines a finite morphism $f : \overline{\mathcal{X}} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$ by Lemma 2.2.

For $y \in (\mathcal{X} - \mathcal{Z})_k$, g is unramified at y by assumption, that is, $d(s_i/s^m)$'s are linearly independent at y. On the other hand, we see that $d(u_i/u_0)$ has the form $(s^{qm}/t)^p d(s_i/s^m)$ modulo p. So $d(u_i/u_0)$'s are also linearly independent at y. Hence f is unramified at y. It is easy to check that f satisfies the conditions (2) and (3) in the statement of the theorem. Hence, to prove the theorem, it suffices to prove the flatness of f, that is, the flatness of f modulo $\pi^m O_K$ for all $m \in \mathbb{N}$.

Note that f modulo πO_K is flat, because $\overline{\mathcal{X}}_k, \mathbb{P}^n_k$ are regular of dimension n and all the fibers of f modulo πO_K is 0-dimensional (see [Gr-D, 6.1]). So it suffices to prove the following claim to finish the proof of the theorem:

claim. Let $f_m : \mathcal{P}_m \longrightarrow \mathcal{Q}_m$ be a finite morphism of flat $O_K/\pi^m O_K$-schemes such that f_m modulo πO_K is flat. Then f_m is flat.

For $\nu \leq m$, let $f_\nu : \mathcal{P}_\nu \longrightarrow \mathcal{Q}_\nu$ be f_m modulo $\pi^\nu O_K$. To prove the claim, we may assume that f_ν is flat for $\nu \leq m-1$ as induction hypothesis. Moreover, since the assertion is local on $\mathcal{Q}_m$, we may assume that $\mathcal{Q}_m = \operatorname{Spec} A_m$ is the spectrum of a local ring. Let us put $\mathcal{Q}_\nu := \operatorname{Spec} A_\nu$. Then $\mathcal{P}_\nu$, being finite over $\mathcal{Q}_\nu$ is also affine (so put $\mathcal{P}_\nu =: \operatorname{Spec} B_\nu$) and since $\mathcal{P}_{m-1}$ is finite flat over $\mathcal{Q}_{m-1}$, there exists some $\mu \in \mathbb{N}$ and an isomorphism $\varphi_{m-1} : A_{m-1}^{\oplus\mu} \xrightarrow{=} B_{m-1}$ as A_{m-1}-modules. Let $\varphi_\nu : A_\nu^{\oplus\mu} \xrightarrow{=} B_\nu$ be φ_m modulo $\nu \, (\nu \leq m-1)$ and let us take a homomorphism $\varphi_m : A_m^{\oplus\mu} \longrightarrow B_m$ as A_m-modules lifting φ_{m-1}. Then we have the following diagram whose horizontal lines are exact by flatness of $\mathcal{P}_m, \mathcal{Q}_m$ over $O_K/\pi^m O_K$:

$$\begin{tikzcd} 0 \arrow[r] & A_{m-1}^{\oplus\mu} \arrow[r, "\pi"] \arrow[d, "\varphi_{m-1}", "\|"'] & A_m^{\oplus\mu} \arrow[r] \arrow[d, "\varphi_m"] & A_1^{\oplus\mu} \arrow[r] \arrow[d, "\varphi_1", "\|"'] & 0 \\ 0 \arrow[r] & B_{m-1} \arrow[r, "\pi"] & B_m \arrow[r] & B_1 \arrow[r] & 0. \end{tikzcd}$$

From this diagram, we see that φ_m is also an isomorphism and so f_m is flat. Hence the claim is proved and so the proof of the theorem is finished. □

Let $(\mathbf{X}, \overline{\mathbf{X}})$ be a smooth pair over O_K, let x be a closed point on $\overline{\mathbf{X}}_k$ and let us put $\mathbf{D} := \overline{\mathbf{X}} - \mathbf{X} = \bigcup_{i=1}^r \mathbf{D}_i$ (where $\mathbf{D}_i$'s are connected and smooth over $\operatorname{Spec} O_K$). Then we can define $I_{\ni x}, I_{\not\ni x}, \mathbf{D}_{\ni x}, \mathbf{D}_{\not\ni x}$ as in the case of formal smooth pair. We say that the smooth pair $(\mathbf{X}, \overline{\mathbf{X}})$ over O_K satisfies the condition $(*)_x$ if, for any $I \subseteq I_{\ni x}$, $\bigcap_{i \in I} \mathbf{D}_i$ is irreducible. With this terminology, we have the following algebraic variant of Theorem 2.1 as a corollary:

Corollary 2.6. *Let $(\mathbf{X}, \overline{\mathbf{X}})$ be a smooth pair over O_K with $\dim(\overline{\mathbf{X}}/O_K) = n$ and $\overline{\mathbf{X}}$ projective over $\operatorname{Spec} O_K$. Let x be a closed point in $\overline{\mathbf{X}}$ and assume that $(\mathbf{X}, \overline{\mathbf{X}})$ satisfies the condition $(*)_x$. Let us put $\mathbf{D} := \overline{\mathbf{X}} - \mathbf{X}$, let us define $\mathbf{D}_{\ni x}, \mathbf{D}_{\not\ni x}$ as above and let $\mathbf{D}_{\ni x} = \bigcup_{i=1}^r \mathbf{D}_i$ be the decomposition of $\mathbf{D}_{\ni x}$ into irreducible components. Then there exists a finite flat morphism $f : \overline{\mathbf{X}} \longrightarrow \mathbb{P}^n_{O_K}$ satisfying the following conditions*:

(1) *f is etale on a neighborhood of $f^{-1}(\mathbb{A}^n_k)$.*

(2) *We have $\mathbf{D}_{\not\ni x} \subseteq f^{-1}(H_0)$ and $x \in f^{-1}(\mathbb{A}^n_{O_K})$, where $H_0 := \mathbb{P}^n_{O_K} - \mathbb{A}^n_{O_K} \subseteq \mathbb{P}^n_{O_K}$ is the hyperplane at infinity.*

(3) *For $1 \leq i \leq r$, we have $\mathbf{D}_i \subseteq f^{-1}(H_i)$, where $H_i \subseteq \mathbb{P}^n_{O_K}$ is the i-th coordinate hyperplane. (Hence we see that $\mathbf{D}_i \cap f^{-1}(V) \subseteq f^{-1}(H_i \cap V)$ is an open and closed immersion if $f^{-1}(V) \longrightarrow V$ is etale.)*

Proof. Since the formal smooth pair $(\widehat{\mathbf{X}}, \widehat{\overline{\mathbf{X}}})$ satisfies the assumption of Theorem 2.1, we have a finite flat morphism $\widehat{f} : \widehat{\overline{\mathbf{X}}} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$ satisfying the conclusion of Theorem 2.1. Note that the morphism $\widehat{f}$ is induced by some sections $u_0, ..., u_n \in \Gamma(\widehat{\overline{\mathbf{X}}}, \mathcal{L})$ for some ample line bundle $\mathcal{L}$ on $\widehat{\overline{\mathbf{X}}}$. Let $\mathbf{L}$ be the algebraization of $\mathcal{L}$, which is an ample line bundle on $\overline{\mathbf{X}}$. Then $u_0, ..., u_n$ can be regarded as sections in $\Gamma(\overline{\mathbf{X}}, \mathbf{L})$ and they do not have common zero. So they induce a morphism $f : \overline{\mathbf{X}} \longrightarrow \widehat{\mathbb{P}}^n_{O_K}$, which is the algebraization of $\widehat{f}$, and we can see that it is finite by the argument of [Ke1, Lemma 3]. Then $f_*\mathcal{O}_{\overline{\mathbf{X}}}$ is a coherent sheaf on $\mathcal{O}_{\mathbb{P}^n_{O_K}}$ such that $f_*\mathcal{O}_{\overline{\mathbf{X}}} \otimes_{\mathcal{O}_{\mathbb{P}^n_{O_K}}} \mathcal{O}_{\widehat{\mathbb{P}}^n_{O_K}} = \widehat{f}_*\mathcal{O}_{\widehat{\overline{\mathbf{X}}}}$ is flat over $\mathcal{O}_{\widehat{\mathbb{P}}^n_{O_K}}$. Hence $f_*\mathcal{O}_{\overline{\mathbf{X}}}$ is flat over $\mathcal{O}_{\mathbb{P}^n_{O_K}}$ by GAGA. So f is a finite flat morphism. We can check that this f satisfies the required properties by using Theorem 2.1 and GAGA. □

We prove one more preliminary lemma before the proof of the second main theorem. Let $(\mathbf{X}, \overline{\mathbf{X}})$ be a smooth pair over O_K and put $\mathbf{D} := \overline{\mathbf{X}} - \mathbf{X} = \bigcup_{i=1}^r \mathbf{D}_i$, where $\mathbf{D}_i$'s are connected and smooth over O_K. Then we call a closed subscheme $\mathbf{C} \hookrightarrow \overline{\mathbf{X}}$ *toric* if, locally on $\overline{\mathbf{X}}$, $\mathbf{C}$ has the form $\bigcap_{i \in I} \mathbf{D}_i$ for some $I \subseteq [1, r]$. A morphism $\varphi : \overline{\mathbf{X}}' \longrightarrow \overline{\mathbf{X}}$ is called *a toric blow-up* if it is a blow-up with toric center, and for $x \in \overline{\mathbf{X}}$, the morphism φ is called *a toric blow-up away from x* if it is a toric blow-up whose center does not contain x. If $\varphi : \overline{\mathbf{X}}' \longrightarrow \overline{\mathbf{X}}$ is a toric blow-up and if we put $\mathbf{X}' := \varphi^{-1}(\mathbf{X})$, it induces the strict morphism of smooth pairs $(\mathbf{X}', \overline{\mathbf{X}}') \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ over O_K, which we also denote by φ. Then we have the following lemma:

Lemma 2.7. *Let $(\mathbf{X}, \overline{\mathbf{X}})$ be a smooth pair over O_K and let x be a closed point of $\overline{\mathbf{X}}_k$. Then there exists a morphism of smooth pairs $\varphi : (\mathbf{X}', \overline{\mathbf{X}}') \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ over O_K which is a compsoition of toric blow-ups away from x such that $(\mathbf{X}', \overline{\mathbf{X}}')$ satisfies $(*)_x$.*

Proof. Let $\mathbf{D} = \bigcup_{i=1}^r \mathbf{D}_i$ and $I_{\ni x}$ as before. Then, for any $I \subseteq I_{\ni x}$, we have the decomposition $\bigcap_{i \in I} \mathbf{D}_i = \mathbf{D}_{I,0} \coprod \mathbf{D}_{I,1}$, where $\mathbf{D}_{I,0}$ is the connected component of $\bigcap_{i\in I} \mathbf{D}_i$ containing x and $\mathbf{D}_{I,1}$ is the union of connected components not containing x. Then, by taking the composition of blow-ups along the (strong transforms of) $\mathbf{D}_{I,1}$'s, we obtain the desired morphism φ. $\square$

Now we give a proof of Theorem 0.2.

Proof of Theorem 0.2. For a closed point x of $\overline{\mathbf{X}}_k$, let us take a morphism of smooth pairs $\varphi : (\mathbf{X}', \overline{\mathbf{X}}') \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ over O_K which is a compsoition of toric blow-ups away from x such that $(\mathbf{X}', \overline{\mathbf{X}}')$ satisfies $(*)_x$. (This is possible by Lemma 2.7.) Let us put $\mathbf{D}' := \overline{\mathbf{X}}' - \mathbf{X}'$ and define $\mathbf{D}'_{\ni x} = \bigcup_{i=1}^r \mathbf{D}'_i, \mathbf{D}'_{\not\ni x} \subseteq \mathbf{D}'$ as before. Then we see that the exceptional divisor of φ is contained in $\mathbf{D}'_{\not\ni x}$. Since $(\mathbf{X}', \overline{\mathbf{X}}')$ satisfies $(*)_x$, we can apply Corollary 2.6 to it: Let $f : \overline{\mathbf{X}}' \longrightarrow \mathbb{P}^n_{O_K}$ be a morphism satisfying the conditions in the statement of Corollary 2.6, and let V be an open subscheme of $\mathbb{A}^n_{O_K}$ containing $\mathbb{A}^n_k$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \longrightarrow V$ is finite etale (which exists by (1) of Corollary 2.6) and let us put $\overline{\mathbf{X}}'_x := f^{-1}(V)$. Then, since we have $\mathbf{D}'_{\not\ni x} \cap \overline{\mathbf{X}}'_x = \emptyset$ by (2) of Corollary 2.6, φ induces the isomorphism $\overline{\mathbf{X}}'_x \xrightarrow{=} \varphi(\overline{\mathbf{X}}'_x)$.

For $1 \leq i \leq r$, let H_i be the intersection of the i-th hyperplane in $\mathbb{P}^n_{O_K}$ with V. Then, by (3) of Corollary 2.6, $\mathbf{D}' \cap \overline{\mathbf{X}}'_x = \mathbf{D}'_{\ni x} \cap \overline{\mathbf{X}}'_x$ is a sub relative simple normal crossing divisor of $\bigcup_{i=1}^r f^{-1}(H_i)$ whose complementary divisor C does not contain x. Now let us put $\mathbf{X}'_x := \mathbf{X}' \cap \overline{\mathbf{X}}'_x$, $\mathbf{X}''_x := \mathbf{X}'_x - C$, $\overline{\mathbf{X}}''_x := \overline{\mathbf{X}}'_x - C$. Then we have the diagram

$$(\mathbf{X}, \overline{\mathbf{X}}) \xleftarrow{\varphi} (\mathbf{X}'_x, \overline{\mathbf{X}}'_x) \xleftarrow{j} (\mathbf{X}''_x, \overline{\mathbf{X}}'_x) \xrightarrow{f} (V - \bigcup_{i=1}^r H_i, V)$$

(where φ, f is the morphism induced by $\varphi : (\mathbf{X}', \overline{\mathbf{X}}') \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$, $f : \overline{\mathbf{X}}' \longrightarrow \mathbb{P}^n_{O_K}$ respectively and j is the canonical inclusion) such that $\varphi \circ j$ induces the isomorphism $(\mathbf{X}''_x, \overline{\mathbf{X}}''_x) \xrightarrow{=} (\mathbf{X} \cap \varphi(\overline{\mathbf{X}}''_x), \varphi(\overline{\mathbf{X}}''_x))$.

First we prove that Theorem 0.2 is reduced to the following claim:

claim. Let (E, ∇) be an object in $\mathrm{MIC}(\mathbf{X}'_{x,K}/K)$ and assume that, for any strict closed immersion $i : (\mathbf{Y}, \overline{\mathbf{Y}}) \hookrightarrow (\mathbf{X}'_x, \overline{\mathbf{X}}'_x)$ with $\dim(\overline{\mathbf{Y}}/O_K) = 1$, $i^*(E, \nabla) \in \mathrm{MIC}(\mathbf{Y}_K/K)$ is overconvergent. Then the restriction of (E, ∇) by j (which is an object in $\mathrm{MIC}(\mathbf{X}''_{x,K}/K)$) is overconvergent.

Indeed, assume the claim is true and assume we are given an object (E, ∇) in $\mathrm{MIC}(\mathbf{X}_K/K)$ satisfying the condition (2) of Theorem 0.1. Then, since $\varphi : (\mathbf{X}'_x, \overline{\mathbf{X}}'_x) \longrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ is a strict open immersion, the restriction of (E, ∇) to $\mathrm{MIC}(\mathbf{X}'_{x,K}/K)$ satisfies the assumption of the claim. Hence the restriction of (E, ∇) by j is overconvergent. So the restriction of (E, ∇) by $(\mathbf{X}''_x, \overline{\mathbf{X}}''_x) = (\mathbf{X} \cap \varphi(\overline{\mathbf{X}}''_x), \varphi(\overline{\mathbf{X}}''_x)) \hookrightarrow (\mathbf{X}, \overline{\mathbf{X}})$ is

also overconvergent. Since this is true for any x, we see that (E, ∇) is overconvergent by Proposition 1.3. So we are reduced to proving the claim above.

In the following, we prove the claim. (So let (E, ∇) be as in the claim.) Note that the p-adic completion of the diagram

$$(\mathbf{X}'_x, \overline{\mathbf{X}}'_x) \xleftarrow{j} (\mathbf{X}''_x, \overline{\mathbf{X}}'_x) \xrightarrow{f} (V - \bigcup_{i=1}^r H_i, V)$$

has the following form:

$$(\widehat{\mathbf{X}}'_x, \widehat{\overline{\mathbf{X}}}'_x) \xleftarrow{\hat{j}} (\widehat{\mathbf{X}}''_x, \widehat{\overline{\mathbf{X}}}'_x) \xrightarrow{\hat{f}} ((V - \bigcup_{i=1}^r H_i)^{\wedge}, \widehat{V}) = (\widehat{\mathbb{G}}^r_{O_K} \times \widehat{\mathbb{A}}^{n-r}_{O_K}, \widehat{\mathbb{A}}^n_{O_K}).$$

For a closed subscheme $y \hookrightarrow \mathbb{G}^{n-1}_{m,O_K}$ which is etale over $\operatorname{Spec} O_K$ and $1 \leq i \leq n$, let $\overline{\mathbf{Y}}_{y,i,0} \hookrightarrow \overline{\mathcal{X}}_0$ be the closed immersion defined by

$$\begin{aligned}\overline{\mathbf{Y}}_{y,i,0} &:= y \times \mathbb{A}^1_{O_K} \hookrightarrow \mathbb{G}^{n-1}_{O_K} \times \mathbb{A}^1_{O_K} \\ &\cong \mathbb{G}^{i-1}_{O_K} \times \mathbb{A}^1_{O_K} \times \mathbb{G}^{n-i}_{O_K} \hookrightarrow \mathbb{A}^n_{O_K}\end{aligned}$$

(where the isomorphism in the second line is the permutation of the i-th factor and the last factor). Then let us put

$$\overline{\mathbf{Y}}_{y,i} := f^{-1}(\overline{\mathbf{Y}}_{y,i,0} \cap V) \supseteq \mathbf{Y}'_{y,i} := \overline{\mathbf{Y}}_{y,i} \cap \mathbf{X}'_x \supseteq \mathbf{Y}''_{y,i} := \overline{\mathbf{Y}}_{y,i} \cap \mathbf{X}''_x.$$

Then it is easy to see that $(\mathbf{Y}''_{y,i}, \overline{\mathbf{Y}}_{y,i})$ is a smooth pair over O_K with $\dim(\overline{\mathbf{Y}}_{y,i}/O_K) = 1$. Then, since the morphism

$$\overline{\mathbf{Y}}_{y,i} - \mathbf{Y}'_{y,i} = \overline{\mathbf{Y}}_{y,i} \cap \mathbf{D}'_i \xrightarrow{\subset} \overline{\mathbf{Y}}_{y,i} \cap f^{-1}(H_i) = \overline{\mathbf{Y}}_{y,i} - \mathbf{Y}''_{y,i}$$

is an open and closed immersion, we see that $(\mathbf{Y}'_{y,i}, \overline{\mathbf{Y}}_{y,i})$ is also a smooth pair over O_K. Hence, by assumption, the restriction of (E, ∇) by $(\mathbf{Y}'_{y,i}, \overline{\mathbf{Y}}_{y,i}) \hookrightarrow (\mathbf{X}'_x, \overline{\mathbf{X}}'_x)$ is overconvergent. So the restriction of (E, ∇) by $(\mathbf{Y}''_{y,i}, \overline{\mathbf{Y}}_{y,i}) \hookrightarrow (\mathbf{X}'_x, \overline{\mathbf{X}}'_x)$ is also overconvergent, that is, the restriction of $(E^{\mathrm{an}}, \nabla^{\mathrm{an}})$ to $\mathrm{MIC}((\widehat{\mathbf{Y}}''_{y,i,K}, \widehat{\overline{\mathbf{Y}}}_{y,i,K})/K)$ is overconvergent for all y and i. Hence, by Remark 1.9, we can conclude that the restriction of $(E^{\mathrm{an}}, \nabla^{\mathrm{an}})$ to $\mathrm{MIC}((\widehat{\mathbf{X}}''_{x,K}, \widehat{\overline{\mathbf{X}}}'_{x,K})/K)$ is overconvergent, that is, the pull-back of (E, ∇) by j is overconvergent. Hence we have proved the claim and so the proof of the theorem is finished. $\square$

References

[Be] P. Berthelot, *Cohomologie rigide et cohomologie rigide à supports propres première partie*, prépublication de l'IRMAR 96-03. Available at http://perso.univ-rennes1.fr/pierre.berthelot/

[Bo-Gu-R] S. Bosch, U. Güntzer and R. Remmert, *Non-Archimedean analysis*, Grundlehren der math. Wissenschaften **261**, Springer-Verlag, 1984.

[Gr-D] A. Grothendieck and J. A. Dieudonné, *Éléments de Géometrie Algébrique IV-1*, Pub. Math. IHES **20**(1964), 5–259.

[Ka] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in Algebraic Analysis, Geometry, and Number Theory, J-I.Igusa ed., 1988, Johns Hopkins University, pp. 191–224.

[Ke1] K. S. Kedlaya, *More étale covers of affine spaces in positive characteristic*, Journal of Algebraic Geometry 14 (2005), 187-192.

[Ke2] K. S. Kedlaya, *Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions*, Compositio Math.. **143**(2007), 1164–1212.

[Ke3] K. S. Kedlaya, *p-adic Differential Equations*, book draft avaiable at http://math.mit.edu/∼kedlaya/papers/index.shtml

[K-X] K. S. Kedlaya and L. Xiao, *Differential modules on p-adic polyannuli*, to appear in Journal de l'Institut de Mathematiques de Jussieu.

[S] A. Shiho, *Cut-by-curves criterion for the log-extendability of overconvergent isocrystals*, preprint.

[T] N. Tsuzuki, *Morphisms of F-isocrystals and the finite monodromy theoerem for unit-root F-isocrystals*, Duke Math J. **111**(2002), 385–419.